ON CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. For certain meromorphic p-valent function $\phi$ and $\psi$, we study a class of function $f(z) = \frac{1}{z} + \sum_{n=p}^{\infty} a_n z^n$, $(a_n \geq 0)$, defined in the punctured unit disc $D$, satisfying $\Re\left\{\frac{(f*\phi)(z)}{(f*\psi)(z)}\right\} > \alpha$ $(0 \leq \alpha \leq 1, z \in D)$. Coefficient estimate distortion theorem and radii of starlikeness and convexity are obtained. Further we consider integral operators associated with functions belonging to the aforementioned class.

1. Introduction

Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n \quad p \in N, N = \{1, 2, 3, \ldots\}$$

which analytic and p-valent punctured unit disk $D = \{z : 0 < |z| < 1\}$. A function $f \in \sum_p$ is said to be in the class $\Omega_p(\alpha)$ of meromorphic p-valently starlike functions of order $\alpha$ in $D$ if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in D; 0 \leq \alpha < p; p \in N)$$

Furthermore, a function $f \in \sum_p$ is said to be in the class $A_p(\alpha)$ of meromorphic p-valently convex function of order $\alpha$ in $D$ if and only if

$$\Re\left\{-\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \alpha \quad (z \in D; 0 \leq \alpha < p; p \in N)$$

The classes $\Omega_p(\alpha)$, $A_p(\alpha)$ and various other subclasses of $\sum_p$ have been studied rather extensively by Frasin and Murugusundaramoorthy [4], Aouf et.al. [1, 2, 3], Joshi and Srivastava [5], Kulkarni et.al. [6], Mogra [8], Owa et.al.[9], Srivastava and Owa [10], Uralegaddi and Somantha [11], and Yang [12].

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The Hadamard product or convolution of the functions \( f(z) \) given by (1) and
\[
g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n
\]
is defined by
\[
(f * g)(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n b_n z^n.
\]
For two functions \( f \) and \( g \) analytic in \( D \), we say that the function \( f \) is subordinate to \( g \) (denoted by \( f \prec g \)) if there exists a Schwarz function \( w(z) \), analytic in \( D \) with \( w(0) = 0 \), and \( |w(z)| < 1 (z \in D) \), such that \( f(z) = g(w(z)) \). We introduce here a class \( M(\alpha, \phi, \psi) \) which is defined as follows: suppose the functions \( \phi(z) \) and \( \psi(z) \) are given by
\[
\phi(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} \lambda_n z^n
\]
\[
\psi(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} \mu_n z^n.
\]
Where \( \mu_n \geq \lambda_n \geq 0 \) \( (\text{for all } n \geq p) \).
A function \( f \in \sum_p \) is said to be in the class \( M(\alpha, \phi, \psi) \) if and only if
\[
\Re \left\{ \frac{(f * \phi)(z)}{(f * \psi)(z)} \right\} > \alpha \quad (0 \leq \alpha < 1, z \in D).
\]
In the next section we derive sufficient conditions for \( f(z) \) to be in the class \( M(\alpha, \phi, \psi) \) and \( \Omega(\alpha, \phi, \psi) \), which are obtained by using coefficient inequalities.

2. COEFFICIENT INEQUALITIES

Theorem 1 If \( f \in \sum_p \) satisfies
\[
\sum_{n=p}^{\infty} \{ k\mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)| \} |a_n| \leq 2(1 - \alpha)
\]
for some \( (0 \leq \alpha < 1) \), and \( (k \geq 1) \), then \( f \in M(\alpha, \phi, \psi) \).

Proof Suppose that (3) holds true for \( \alpha \) \( (0 \leq \alpha < 1) \) and \( \mu_n \geq \lambda_n \) \( (k \geq 1) \), for \( f \in \sum_p \) it suffices to show that
\[
\left| \frac{(f * \phi)(z)}{(f * \psi)(z)} - k \right| < 1 \quad (z \in D)
\]
we note that
\[
\left| \frac{(f * \phi)(z)}{(f * \psi)(z)} - k \right| = \left| \frac{k - 1 + \sum_{n=p}^{\infty} (k\mu_n - \lambda_n) a_n z^{p+n}}{1 - 2\alpha + k + \sum_{n=p}^{\infty} (\lambda_n + \mu_n (k - 2\alpha))|a_n| z^{p+n}} \right|
\]
\[
\leq \left| \frac{k - 1 + \sum_{n=p}^{\infty} (k\mu_n - \lambda_n) |a_n| z^{p+n}}{1 - 2\alpha + k - \sum_{n=p}^{\infty} |\lambda_n + \mu_n (k - 2\alpha)| |a_n| z^{p+n}} \right|
\]
\[
< \left| \frac{k - 1 + \sum_{n=p}^{\infty} (k\mu_n - \lambda_n) |a_n|}{1 - 2\alpha + k - \sum_{n=p}^{\infty} |\lambda_n + \mu_n (k - 2\alpha)| |a_n|} \right|.
\]
In view of Theorem 1, we now define the subclass $\Omega(\alpha, \phi, \psi)$. Since $f(\alpha) < \phi$, we observe that the sequences $\lambda_n = \frac{n}{p}$ and $\mu_n = 1$, which is equivalent to our condition (3) of Theorem 1.

Proof

Let $\phi(z) = \frac{1}{z^p} - \frac{z^p}{(1-z)^2}$ and $\psi(z) = \frac{1}{z^2} + \frac{z^p}{1-z}$ we have $\lambda_n = \frac{n}{p}$ and $\mu_n = 1$, therefore $M(\alpha, \phi, \psi)$ reduce to the class $\Omega_p(\alpha)$. Putting $\lambda_n = \frac{n}{p}$ and $\mu_n = 1$ in Theorem 1, we obtain the following corollary due to Frasin and Murugusundaramoorthy [4].

**Corollary 1** Let $\sigma_n(p, k, \alpha) = (p + n + k - 1) + |p + n + 2\alpha - k - 1|$. If $f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n+p-1}z^{n+p-1}$, satisfies $\sigma_n(p, k, \alpha) |a_{p+n-1}| < 2 (p - \alpha)$ for some $\alpha (0 \leq \alpha < p)$ and some $k (k \geq p)$, then $f(z) \in \Omega_p(\alpha)$.

When $k = 1$ and $\lambda_n + \mu_n (1 - 2\alpha) \leq 0 \leq \mu_n - \lambda_n$ we get the following corollary due to Kumar et al [7].

**Corollary 2** Let $f(z) \in \sum_p$ given by (1). If $\sum_{n=1}^{\infty} (\alpha \mu_n - \lambda_n) |a_n| \leq 1 - \alpha$ for $\alpha (0 \leq \alpha < 1)$, then $f(z) \in M_p(g, h, \alpha)$.

In view of Theorem 1, we now define the subclass $\Omega(\alpha, \phi, \psi)$ of $M(\alpha, \phi, \psi)$ which consists of functions $f(z) \in \sum_p$ satisfying condition (3).

3. DISTORTION THEOREM

**Theorem 2** If the function $f(z)$ defined by (1) is in the class $\Omega(\alpha, \phi, \psi)$, then for $0 \leq |z| = r < 1$, we have

$$
\frac{1}{r^p} - \frac{2 (1-\alpha)}{k \mu_p - \lambda_p + |\lambda_p + \mu_p (k - 2\alpha)|} r^p \leq |f(z)| \leq \frac{1}{r^p} + \frac{2 (1-\alpha)}{k \mu_p - \lambda_p + |\lambda_p + \mu_p (k - 2\alpha)|} r^p,
$$

where the sequence $< \mu_n >$ and $< \mu_n/\lambda_n >$ are nondecreasing.

The bound in (4) is attained for the functions $f(z)$ given by

$$
f(z) = \frac{1}{z^p} + \frac{2 (1-\alpha)}{k \mu_p - \lambda_p + |\lambda_p + \mu_p (k - 2\alpha)|} z^p.
$$

**Proof** We observe that the sequences $< k \mu_n - \lambda_n >$ and $< \lambda_n + \mu_n (k - 2\alpha) >$ are nondecreasing. Since $f(z) \in \Omega(\alpha, \phi, \psi)$ from inequality (3) we have

$$
\sum_{n=1}^{\infty} |a_n| \leq \frac{2 (1-\alpha)}{k \mu_p - \lambda_p + |\lambda_p + \mu_p (k - 2\alpha)|}
$$

Thus for $0 \leq |z| = r < 1$, and making use of (6) we have

$$
|f(z)| \leq \left| \frac{1}{z^p} \right| + \sum_{n=1}^{\infty} |a_n| |z|^n
$$

$$
\leq \frac{1}{r^p} + r^p \sum_{n=1}^{\infty} |a_n|
$$

$$
\leq \frac{1}{r^p} + \frac{2 (1-\alpha)}{k \mu_p - \lambda_p + |\lambda_p + \mu_p (k - 2\alpha)|} r^p
$$
and

\[ |f(z)| \geq \left| \frac{1}{z^p} \right| - \sum_{n=p}^{\infty} |a_n| |z|^n \]

\[ \geq \frac{1}{r^p} - \frac{r^p}{r^p} \sum_{n=p}^{\infty} |a_n| \]

\[ \geq \frac{1}{r^p} - \frac{2 (1 - \alpha)}{\lambda + k (k - 2) \mu (k - 2)} r^p \]

which readily yields the inequality (4). This completes the proof of Theorem 2.

4. RADII OF STARRIKNESS AND CONVEXITY

The radii of starlikeness and convexity for class $M(\alpha, \phi, \psi)$ is given by.

**Theorem 3** If the function $f(z)$ defined by (1) is in the class $M(\alpha, \phi, \psi)$, then $f(z)$ is meromorphically p-valently starlike of order $\delta (0 \leq \delta < 1)$ in $|z| < r_1$ where

\[ r_1 = \inf \left\{ \left( p - \delta \right) [k \mu + \lambda] + \left| \lambda + \mu \left( k - 2 \right) \right| ] \right\}^{\frac{1}{1 - \delta}} \]

(9)

Furthermore, $f(z)$ is meromorphically p-valently convex of order $\delta (0 \leq \delta < 1)$ in $|z| < r_2$ where

\[ r_2 = \inf \left\{ \left( p - \delta - 2 \right) [k \mu + \lambda] + \left| \lambda + \mu \left( k - 2 \right) \right| ] \right\}^{\frac{1}{1 - \delta}} \]

(10)

The results (9), (10) are sharp for the function $f(z)$ given by

\[ f(z) = \frac{1}{z^p} + \frac{2 (k - \alpha)}{k \mu + \lambda} z^{n + p} \]

(11)

**Proof** To prove (9) it suffices to show that

\[ \left| \frac{zf'(z)}{f(z)} + p \right| \leq p - \delta \]

(12)

for $|z| < r_1$ we have

\[ \left| \frac{zf'(z)}{f(z)} + p \right| = \left| \sum_{n=p}^{\infty} (n + p) a_n z^{n + p} \right| \]

\[ \leq \sum_{n=p}^{\infty} (n + p) |a_n| |z|^{n + p} \]

(13)

Hence (13) holds true if

\[ \sum_{n=p}^{\infty} (n + p) |a_n| |z|^{n + p} \leq (p - \delta) \left[ 1 - \sum_{n=p}^{\infty} |a_n| |z|^{n + p} \right] \]

(14)

that is

\[ \sum_{n=p}^{\infty} \frac{(n + 2p - \delta)}{(p - \delta)} |a_n| |z|^{n + p} \leq 1. \]

(15)
With the aid of (3), (15) is true if
\[ |z|^{n+p} \left( \frac{n + 2p - \delta}{p - \delta} \right) \leq \frac{k\mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)|}{2(1 - \alpha)} \quad n \geq 1. \] (16)

Solving (16) for $|z|$ we obtain
\[ |z| \leq \left( \frac{(p - \delta) \{k\mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)|\}}{2(n + 2p - \delta) (1 - \alpha)} \right)^{\frac{1}{n+p}} \quad n \geq 1. \] (17)

In precisely the same manner, we can find the radius of convexity asserted by (10) by requiring that
\[ \left| \frac{z f''(z)}{f'(z)} + p + 1 \right| \leq p - \delta. \] (18)

5. CLOSURE THEOREM

Let the functions $f_j(z)$ be defined, for $j \in \{1, 2, 3, ..., m\}$, by
\[ f_j(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_{n,j} z^n, \quad (z \in D). \] (19)

Now, we shall prove the following result for closure of functions in the class $\Omega(\alpha, \phi, \psi)$.

**Theorem 4** Let the functions $f_j(z)$, $j \in \{1, 2, 3, ..., m\}$, defined by (19) be in the class $\Omega(\alpha, \phi, \psi)$. Then the function $h(z) \in \Omega(\alpha, \phi, \psi)$ where
\[ h(z) = \sum_{j=p}^{m} b_j f_j(z), \quad b_j \geq 0 \text{ and } \sum_{j=p}^{m} b_j = 1. \] (20)

**Proof** From (20), we can write
\[ h(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} c_n z^n, \] (21)

where
\[ c_n = \sum_{j=p}^{m} b_j a_{n,j}, \quad j \in \{1, 2, 3, ..., m\}. \] (22)

Since $f_j(z) \in \Omega(\alpha, \phi, \psi)$, $j \in \{1, 2, 3, ..., m\}$, from (3), we have
\[
\sum_{n=p}^{\infty} \left( k\mu_n - \lambda_n \right) + |\lambda_n + \mu_n (k - 2\alpha)| \left( \sum_{j=p}^{m} b_j a_{n,j} \right) = \sum_{j=p}^{m} b_j \left( \sum_{n=p}^{\infty} \left( k\mu_n - \lambda_n \right) + |\lambda_n + \mu_n (k - 2\alpha)| a_{n,j} \right) \leq \sum_{j=p}^{m} b_j = 1.
\]

Which shows that $h(z) \in \Omega(\alpha, \phi, \psi)$. This completes the proof of Theorem 4.

**Theorem 5** Let
\[ f_{p-1} = \frac{1}{z^p} \quad (z \in D) \] (23)
and
\[ f_n(z) = \frac{1}{z^p} + \frac{(k\mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)|}{2(1 - \alpha)} z^n, \]  
where \( n \geq p; z \in D. \) Then \( f(z) \in \Omega(\alpha, \phi, \psi) \) if and only if it can be expressed in the form
\[ f(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z), \]  
(25)
where \( \lambda_n \geq 0, (n \in \mathbb{N}_0) \) and \( \sum_{n=p-1}^{\infty} \lambda_n = 1. \)

Proof From (23), (24), and (25) it is easily see that
\[ f(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z) \]  
(26)
Since
\[ \sum_{n=p}^{\infty} \frac{(k\mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)|}{2(1 - \alpha)} \]  
\[ \lambda_n = \sum_{n=p}^{\infty} \lambda_n = 1 - \lambda_{p-1} \leq 1, \]
it follows from Theorem 1 that the function \( f(z) \) given by (25) is in the class \( \Omega(\alpha, \phi, \psi) \). Conversely, let us suppose that \( f(z) \in \Omega(\alpha, \phi, \psi) \). Since
\[ |a_n| \leq \frac{2(1 - \alpha)}{(k\mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)|} (n \geq p). \]
Setting
\[ \lambda_n = \frac{(k\mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)|}{2(1 - \alpha)} |a_n| \]  
(27)
and
\[ \lambda_{p-1} = 1 - \sum_{n=p}^{\infty} \lambda_n. \]
It follows that
\[ f(z) = \sum_{n=p-1}^{\infty} \lambda_n f_n(z). \]
This completes the proof of the Theorem 5.

6. INTEGRAL OPERATOR

Theorem 6 Let the function \( f(z) \) defined by (1) be in the class \( \Omega(\alpha, \phi, \psi) \) and \( c > p \) be a real number such that \( c > p \). Then the function \( F(z) \) defined by
\[ F(z) = \frac{c-p}{z^c} \int_0^z t^{c-1} f(t) dt \]  
(27)
also belongs to the class \( \Omega(\alpha, \lambda, g) \).
Proof From the representation of \( F(z) \), it follows that
\[ F(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} c_n z^n, \]  
(28)
where 

\[ c_n = \left( \frac{c - p}{c + n} \right) a_n. \] \hspace{1cm} (29)

Therefore

\[ \sum_{n=p}^{\infty} |k \mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)|| c_n \]

\[ = \sum_{n=p}^{\infty} |k \mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)|| \left( \frac{c - 1}{c + n} \right) a_n \] \hspace{1cm} (30)

\[ \leq \sum_{n=2}^{\infty} |k \mu_n - \lambda_n + |\lambda_n + \mu_n (k - 2\alpha)|| a_n \leq 2 (1 - \alpha) \]

since \( f(z) \in \Omega (\alpha, \phi, \psi) \). Hence, by Theorem 1, \( F(z) \in \Omega (\alpha, \phi, \psi) \).

**Theorem 7** Let \( c \) be a real number such that \( c > p \). If \( F(z) \in \Omega (\alpha, \phi, \psi) \). Then the function \( f(z) \) defined by (27) is meromorphically \( p \)-valent close-to-convex in |\( z \)| < \( r^* \), where

\[ r^* = \inf_n \left\{ \frac{(p - \alpha)(c - p) |(k \mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)||}{2n(c + n)(1 - \alpha)} \right\}^{1/p} \hspace{1cm} (n \geq p) \] \hspace{1cm} (31)

The result is sharp .

**Proof** Let \( F(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} a_n z^n (a_n \geq 0) \). it follows from (27) that

\[ f(z) = z^{1-c} [z^c F(z)]' = \frac{1}{z^p} + \sum_{n=p}^{\infty} \left( \frac{c + n}{c - p} \right) a_n z^n \hspace{1cm} (c > p). \] \hspace{1cm} (32)

In order to obtain the required result it suffices to show that \( |z^{p+1} f'(z) + p| \leq p - \alpha \) in |\( z \)| < \( r^* \). Now

\[ |z^{p+1} f'(z) + p| \leq \sum_{n=p}^{\infty} \frac{n(c + n)}{(c - p)} a_n |z|^{n-1}. \]

Thus \( |z^{p+1} f'(z) + p| < p - \alpha \) if

\[ \sum_{n=p}^{\infty} \frac{n(c + n)}{(p - \alpha)(c - p)} a_n |z|^{n-1} < 1. \] \hspace{1cm} (33)

Hence by using (3), (33) will be satisfied if

\[ \frac{n(c + n)}{(p - \alpha)(c - p)} |z|^{n+p} < \frac{(k \mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)||}{2(1 - \alpha)}, \]

i.e., if

\[ |z| < \left[ \frac{(p - \alpha)(c - p) |(k \mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)||}{2n(c + n)(1 - \alpha)} \right]^{1/p} \hspace{1cm} (n \geq p). \] \hspace{1cm} (34)

Therefore \( f(z) \) is meromorphic close-to-convex in |\( z \)| < \( r^* \). Sharpness follows if we take

\[ f(z) = z + \frac{2n(c + n)(1 - \alpha)}{(p - \alpha)(c - p) |(k \mu_n - \lambda_n) + |\lambda_n + \mu_n (k - 2\alpha)||} z^n \] \hspace{1cm} (n \geq p, c > p).

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