NUMERICAL SOLUTION OF FREDHOLM-VOLTERA
FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH
NONLOCAL BOUNDARY CONDITIONS

AMIT SETIA, YUCHENG LIU, A.S. VATSALA

Abstract. In this paper, a numerical method is proposed to solve Fredholm-
Volterra fractional integro-differential equation with nonlocal boundary con-
ditions. For this purpose, the Chebyshev wavelets of second kind are used in
collocation method. It reduces the given fractional integro-differential equa-
tion (FIDE) with nonlocal boundary conditions in a linear system of equations
which one can solve easily. The test examples are taken from the literature
in order to illustrate the proposed method and different comparisons are also
shown. The involved errors are measured with RMS-norm to show the accu-
racyn obtained.

1. Introduction

Qualitative and numerical studies of fractional differential equations have played
an important role due to its application [1]. Computation of the solution of frac-
tional differential [2] has proved to be challenging since we do not have separation
of variable method and product rules which are readily available for integer order
differential equations. Since there is no general method like in case of integer order
differential equations. Many researchers in the field of fractional differential equa-
tions and fractional integro-differential equations have proposed different methods
to compute the solution. A few among these are transform method [3], homo-
topy analysis [4], Adomian decomposition method [5], variational iteration method
[6], homotopy perturbation method [6] and collocation method [7, 8]. But im-
plementation of wavelet based methods [9, 10, 11] for solving fractional differential
equations have started recently. Further some researchers have also worked on frac-
tional integro-differential equation by using wavelet based methods [12, 13, 14, 15].
In this paper, we have proposed a numerical method to solve the most general
Fredholm-Volterra fractional integro-differential equation (1) with non-local bound-
ary conditions (2) from [16] by using Chebyshev wavelets of second kind.

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functions, Nonlocal boundary conditions.
Table 1. Comparison of recent methods [12, 13] with our proposed method in this paper,

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<td>Yes</td>
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<td>discussion for noisy data</td>
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\[
D^q u(x) = g(x) + f(x)u(x) + \int_a^x k_1(x, t)u(t)dt + \int_a^b k_2(x, t)u(t)dt \tag{1}
\]

\[
\sum_{j=1}^m (\alpha_{ij}u^{(j-1)}(a) + \beta_{ij}u^{(j-1)}(b)) + \mu_i \int_a^b H_i(t)u(t)dt = c_i, i = 1, 2, ..., m. \tag{2}
\]

where \( m-1 < q \leq m, a < x, t < b \) and \( m \in N \), \( D^q \) denotes a differential operator with fractional order \( q \); \( f(x), g(x) \) and \( k_i(x, t), i = 1, 2, \) are analytic functions, \( H_i(t) \) is a continuous function, \( \alpha_{ij}, \beta_{ij}, \mu_i \) and \( c_i, (i = 1, 2, ..., m) \) are constants and \( u(x) \) is a function of class \( C^m \) (functions which are continuously differentiable of order \( m \) on \( I = (0, \infty) \) and integrable on any finite subinterval of \( J = [0, \infty) \)).

Our work is motivated from very recent work in the field of fractional integro-differential equations involving nonlocal boundary conditions [16, 17], numerical methods based on wavelets [9, 10, 11, 12, 13, 14, 15] and one of an interesting application [11]. To the best of our knowledge, nobody has solved a general problem (1) of Fredholm-Volterra type fractional integro-differential equation with nonlocal boundary conditions (2) by using any of wavelet based method. Further, in order to conclude the originality of our work, we have compared our method with the recently proposed methods [12, 13] in the Table 1.

2. Basic definition of Chebyshev wavelets

The Chebyshev wavelets of second kind can be defined as

\[
\psi_{n,l}(x) = \begin{cases} \sqrt{\frac{2^n}{\pi}} T_l(2^k x - (2n - 1)), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{elsewhere} \end{cases} \tag{3}
\]

where \( n = 1, 2, ..., 2^{k-1} \); \( k \) can assume any positive integer; \( T_l(x) \) are second kind Chebyshev polynomials of degree \( l \).

The second kind Chebyshev polynomials \( T_l(x) \) are defined by

\[
T_0(x) = 1, T_1(x) = 2x, T_{l+1}(x) = 2xT_l(x) - T_{l-1}(x) \tag{4}
\]

where \( l = 1, 2, ... \).
These are orthogonal polynomials with respect to the weight function \( w(x) = \sqrt{1-x^2} \) on \([-1,1]\). The dilated and translated weight function can be defined as
\[
w_{kn}(x) = w(2kx - 2n + 1)
\]
for a given value of \( k \) and \( n \) appearing in second kind Chebyshev wavelet.

A function \( u(x) \in L^2_{u_{kn}}[0,1] \) is expanded as
\[
u(x) = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} c_{n,l} \psi_{n,l}(x) \tag{6}\]
where \( c_{n,l} = \langle u(x), \psi_{n,l}(x) \rangle \) in which \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2_{u_{kn}}[0,1] \).

The truncated series of (6) can be defined as
\[
u(x) = \sum_{n=1}^{2k-1} \sum_{l=0}^{R-1} c_{n,l} \psi_{n,l}(x) = C^T \psi(x), \tag{7}\]
where \( C = [c_1, c_2, ..., c_{2k-1}]^T, \psi(x) = [\psi_1, \psi_2, ..., \psi_{2k-1}]^T, c_i = [c_{i,0}, c_{i,1}, ..., c_{i,R-1}]^T, \psi_i = [\psi_{i,0}, \psi_{i,1}, ..., \psi_{i,R-1}]^T \) for \( i = 1, 2, ..., 2^k-1 \).

The definitions and properties of Riemann-Liouville fractional integral operator of a given order and Caputo definition of fractional differential operator have been followed from [2].

3. Methodology for the solution

Let \( D^q u(x) = C^T \psi(x), m - 1 < q \leq m \),
\[
\tag{8}
\]
where \( C \) is defined as in (7).

This gives \( u(x) = C^T P^q \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{x^k}{k!}, m - 1 < q \leq m \),
\[
\tag{9}
\]
where \( u^{(k)}(x) \) denotes the \( k \)th derivative of function \( u(x) \) and the matrix \( P \) is defined as the operational matrix of the fractional integration [18].

Let \( f_1(x) \approx F_1^T \psi(x) \),
\[
\tag{10}
\]
where \( F_1 \) is a coefficient matrix defined similar as in (7).

\[
f_2(x)u(x) \approx f_2(x) \left[ C^T P^q \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{x^k}{k!} \right] \tag{11}\]

\[
\int_a^x k_1(x,t)u(t)dt \approx \int_a^x \left[ C^T P^q \psi(t) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!} \right] \left[ \psi(t)^T K_1 \psi(x) \right] dt
\]
\[
= C^T P^q L_x K_1 \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) L_x^k K_1 \psi(x) \tag{12}\]
where
\[ \int_a^x \psi(t)\psi(t)^T dt = L_x \]
\[ \int_a^x \frac{t^0}{0!} \psi(t)^T dt = L_x^0 \]
\[ \int_a^x \frac{t^1}{1!} \psi(t)^T dt = L_x^1 \]
\[ \vdots \]
\[ \int_a^x \frac{t^{m-1}}{(m-1)!} \psi(t)^T dt = L_x^{m-1} \]

\[ \int_a^1 k_2(x,t)u(t)dt \approx \int_a^1 \left[ C^T P^q \psi(t) + \sum_{k=0}^{m-1} \left[ u^{(k)}(0) \frac{t^k}{k!} \right] \right] \psi(t)^T K_2 \psi(x) dt \]
\[ = C^T P^q M K_2 \psi(x) + \sum_{k=0}^{m-1} \left[ u^{(k)}(0) M^k K_2 \psi(x) \right] \] (14)

where
\[ \int_a^1 \psi(t)\psi(t)^T dt = M \]
\[ \int_a^1 \frac{t^0}{0!} \psi(t)^T dt = M^0 \]
\[ \int_a^1 \frac{t^1}{1!} \psi(t)^T dt = M^1 \]
\[ \vdots \]
\[ \int_a^1 \frac{t^{m-1}}{(m-1)!} \psi(t)^T dt = M^{m-1} \]

So using the equations (7) to (14), the equation (1) becomes
\[ C^T \psi(x) = F_1^T \psi(x) + f_2(x) C^T P^q \psi(x) + f_2(x) \left[ \sum_{k=0}^{m-1} \left[ u^{(k)}(0) \frac{x^k}{k!} \right] \right] \]
\[ + C^T P^q L_2 K_1 \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) L_2^k K_1 \psi(x) \]
\[ + C^T P^q M K_2 \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) M^k K_2 \psi(x) \].
Choosing $x_i = \frac{2i-1}{2\omega}$, $i = 1, 2, ..., \omega = 2^{k-1}R$, collocation points between 0 and 1, one gets

$$C^T \psi(x_i) = F_1^T \psi(x_i) + f_2(x_i)C^T P^q \psi(x_i) + f_2(x_i) \left[ \sum_{k=0}^{m-1} u^{(k)}(0) \frac{x_i^k}{k!} \right]$$

$$+ C^T P^q \sum_{k=0}^{m-1} u^{(k)}(0) L^k \psi(x_i)$$

$$+ C^T P^q \sum_{k=0}^{m-1} u^{(k)}(0) M^k \psi(x_i),$$

where $i = 1, 2, ..., \omega$.

Now finding the $r^{th}$ derivative of $u(x)$ as

$$u^{(r)}(x) = D^r u(x) = C^T P^q \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!}$$

$$= C^T P^q d_i + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!}$$

where

$$\int_a^b H_i(t) t^k dt = d_i$$

$$\int_a^b \frac{t^0}{0!} H_i(t) dt = d_i^0$$

$$\int_a^b \frac{t^1}{1!} H_i(t) dt = d_i^1$$

$$\vdots$$

$$\int_a^b \frac{t^{m-1}}{(m-1)!} H_i(t) dt = d_i^{(m-1)}$$

So using equations (17) to (19), the equation (2) becomes

$$\sum_{j=1}^{m} \left( \alpha_{ij} [C^T P^q \psi(0)] + \beta_{ij} [C^T P^q \psi(1)] \right) + \mu_i \left( C^T P^q d_i + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!} \right) = c_i$$

where $i = 1, 2, ..., m$.

Hence (16) and (20) form a system of $\omega + m$ equations in $\omega + m$ unknowns which one can easily solve.

In the next section, the proposed method is implemented for the test examples corresponding to fractional integro-differential equations (1) with nonlocal boundary conditions (2). The comparisons are also shown with the results in [16].
4. Illustrative Examples

In equation (1), the exact data function is denoted by \( g(x) \). And, one can obtain a noisy data function by adding a random error \( \epsilon \theta_i \) to \( g(x) \), i.e. \( g'(x_i) = g(x_i) + \epsilon \theta_i \) where \( x_i = ih; i = 0, 1, 2, ..., N; Nh = 1 \) and \( \theta_i \) is a uniform random variable with values in \([-1, 1]\) such that \( \max_{1 \leq i \leq N} |g_e(x_i) - g(x_i)| \leq \epsilon \). The accuracy of the proposed method is further measured with the help of absolute errors and discrete \( L_2 \)-norm of absolute errors. The discrete \( L_2 \)-norm of absolute errors can be defined as \( \left( \frac{1}{N+1} \sum_{i=0}^{N} |u_{\text{exact}}(x_i) - u(x_i)|^2 \right)^{1/2} \). This discrete \( L_2 \)-norm of absolute errors is also known as root mean square error norm (i.e. RMS-norm). Further \( E_0, E_1 \) and \( E_2 \) are defined as the random fluctuations \( u_e(x_i) - u_{\text{exact}}(x_i), \) of the solution \( u(x) \). These occur when random noises are introduced in the given data function \( g(x) \). The random noises are \( \epsilon = 0, 0.001 \) and 0.01, respectively for \( E_0, E_1 \) and \( E_2 \).

Theorem 1(a) Assume \( \eta \) be the number of vanishing moments for a wavelet \( \psi_{n,l}(x) \) and let \( f(x) \in C^n[0, 1] \). Then the wavelet coefficient \( c_{n,l} \) decays as follows:

\[
|c_{n,l}| \leq C_\eta 2^{-\eta(n+\frac{1}{2})} \max_{\xi \in [0,1]} |f^n(\xi)|
\]

where \( C_\eta \) is an independent constant from \( n, l \) and \( f(x) \).

(b) Suppose \( f(x) \in C^n[0, 1] \) and \( f(x) \approx \sum_{n=1}^{2^k-1} \sum_{l=0}^{R-1} c_{n,l} \psi_{n,l}(x) = C^T \psi(x) \) is the approximate solution by using Chebyshev wavelets method. Then the error bound is obtained as follows:

\[
||error(f(x)|| \leq \frac{1}{\eta^{2^k-1}} \max_{\xi \in [0,1]} |f^n(\xi)|
\]

where \( error(f(x)) = f(x) - C^T \psi(x) \).

Proof: For details, one can refer [19].

Remark: Importance of Theorem 1.

Theorem 1(a) implies that wavelet coefficients are exponentially decayed with respect to \( \eta \) and by increasing \( \eta \) the decay increases while Theorem 1(b) provides an error bound in a function approximation by Chebyshev wavelet of second kind. One can notice that \( f(x) \) and \( g(x) \) (with reference to equation (1)) in all the examples, belong to a class of infinitely differentiable continuous functions \( C^\infty[0, 1] \). So the above theorem concludes that the wavelet coefficients are going to be very small and a good approximation can be obtained easily.

Example 4.1 Consider the following fractional integro-differential equation from [19]:

\[
D^{1/2}u(x) = \frac{\sqrt{x}}{\Gamma(3/2)} - \frac{x^2}{2} - x^2 e^x u(x) + e^x \int_0^x t u(t) dt + \int_0^1 x^2 u(t) dt.
\]

The nonlocal boundary condition is given by

\[
u(0) + u(1) - 3 \int_0^1 t u(t) dt = 0.
\]

The Example 4.1 has an exact solution \( u(x) = x \). For the problem defined in Example 4.1, the absolute errors are compared by using different basis functions. Like the basis functions in matrix form for \( k = 2, R = 2 \), are given by \( \psi(x) = [\psi_0, \psi_1, \psi_2, \psi_3]^T \). The numerical results are shown in Figure 1. It is observed that as the value of \( k \) or \( R \) or both increases, the absolute error decreases. From the Figure 1, it can be noticed that the maximum absolute error occurs for \( k = 2, R = 2 \),
while the least error occurs for $k = 4, R = 2$. And, the absolute errors are also compared with [16].

**Example 4.2** Consider the following fractional integro-differential equation from [16]:

\[ D^{5/4}u(x) = \frac{8}{3} \frac{x^{3/4}}{\Gamma(3/4)} + 2 - 2 \cos x + x^2 \sin x - 2x \sin x + [\cos x - \sin x]u(x) + \int_0^x \sin t u(t) dt. \]

The nonlocal boundary conditions are given as follows:

\[ u(0) + u(1) + \left( \frac{e + 1}{e + 2} \right) u'(0) + \frac{1}{2} u'(1) - \int_0^1 t u(t) dt = 0, \]

\[ 2u(0) + 2u(1) + \left( \frac{e}{e + 1} \right) u'(0) - u'(1) = 0. \]

Further, the Example 4.2 has an exact solution $u(x) = x^2$. And, the numerical results for absolute errors by using different pair of values of $k$ and $R$ (like $k =$
Comparison of absolute errors

Example 4.3 Let the fractional integro-differential equation from [16] be:

\[ D^{1/3}u(x) = \frac{3}{2} \frac{x^{2/3}}{\Gamma(2/3)} - 1 + e^{x^2} - x^2 e^{x^2} + \int_0^x x^2 e^{x^1} u(t)dt. \]

The nonlocal boundary condition is

\[ u(0) + 2u(1) + 3 \int_0^1 tu(t)dt = 3. \]

Example 4.3 has an exact solution \( u(x) = x \). Figure 3 compares the absolute errors when different basis functions are used in Example 4.3. The results are also
Figure 5. compares the errors $E_0$, $E_1$ and $E_2$ for Example 4.1.

Figure 6. compares the errors $E_0$, $E_1$ and $E_2$ for Example 4.2.

Figure 7. compares the errors $E_0$, $E_1$ and $E_2$ for Example 4.3.

compared with those obtained in [16]. And, one can easily observe that a desired
Table 2. It measures errors with the help of RMS-norm for $k = 2, R = 2; N = 50.$

<table>
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<th>$E_0$</th>
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<td>0.0104</td>
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<td>0.0203</td>
<td>0.0204</td>
</tr>
<tr>
<td>Example 4.3</td>
<td>0.0088</td>
<td>0.0087</td>
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Accuracy can be obtained by choosing higher values of $k$ and $R$. Figure 4 compares the errors measured with RMS-norm. It also shows the comparison among all the three examples by using different basis functions. Figures 5, 6 and 7 compare the random fluctuations $E_0, E_1$ and $E_2$ for Example 4.1, Example 4.2 and Example 4.3, respectively. It shows that on introducing a small random noise in the given data, the random fluctuations also remain small. It concludes that the proposed numerical solutions are numerically stable with respect to noisy data. In practical situations, it may not be possible all the times to find the exact data or sometimes it may not be economical to obtain it. Then the numerical stability result is of great importance.

In Table 2, the errors $E_0, E_1, E_2$ are measured by using RMS-norm for $N = 50$. A particular case of basis functions (i.e. $k = 2, R = 2$), is used and the results are compared for all the three examples. It is observed from Table 2 that the least random fluctuation occurs in Example 4.3. This shows that the problem described in Example 4.3 is more stable than the problems described in other two examples.

5. Conclusions

A numerical method is proposed for solving the most general Fredholm-Volterra fractional integro-differential equation with non-local boundary conditions by using Chebyshev wavelet of second kind. The accuracy of proposed algorithm is illustrated by calculating the parameters like absolute error and RMS-error as in [20]. Figure 2(b) clearly shows that as the basis functions increase, the RMS-errors decrease very fast and a given accuracy can be obtained by choosing the suitable number of basis functions. Table 2 confirms the numerical stability of the proposed algorithm even when given data function is very noisy.

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