VOLterra-Stieltjes integral equation in reflexive Banach spaces

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Abstract. Volterra-Stieltjes integral equations have been studied in the space of continuous functions in many papers for example, (see [3]-[7]). Our aim here is to studying the existence of weak solutions to a nonlinear integral equation of Volterra-Stieltjes type in a reflexive Banach space. A special case will be considered.

1. Introduction and Preliminaries

Let $E$ be a reflexive Banach space with norm $\|\cdot\|$ and dual $E^*$. Denote by $C[I,E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.

Consider the nonlinear Riemann-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \, ds \, g(t, s), \quad t \in I = [0, T],$$

(1)

This type of equations have been studied by Banaś (see [11]-[13]) and also by some other authors, for example (see [7], [9] and [15]-[17]).

Here, we study the existence of a weak solution $x \in C[I,E]$ in the reflexive Banach space $E$ for the nonlinear Volterra-Stieltjes integral equation (1) where $f$ is assumed to be weakly-weakly continuous.

For the properties of the Stieltjes integral (see Banaś [1]).

Now, we shall present some auxiliary results that will be need in this work. Let $E$ be a Banach space (need not be reflexive) and let $x : [a, b] \to E$, then

(1-) $x(\cdot)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(\cdot))$ is continuous (measurable) at $t_0$.

(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if $h$ maps weakly convergent sequences in $E$ to weakly convergent sequences in $E$.

If $x$ is weakly continuous on $I$, then $x$ is strongly measurable and hence weakly measurable (see [14] and [11]). It is evident that in reflexive Banach spaces, if $x$ is

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weakly continuous function on \([a, b]\), then \(x\) is weakly Riemann integrable (see [14]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of equation (1) in the space \(C[I, E]\).

**Definition 1.**

Let \(f : I \times E \rightarrow E\). Then \(f(t, u)\) is said to be weakly-weakly continuous at \((t_0, u_0)\) if given \(\epsilon > 0\), \(\phi \in E^*\) there exists \(\delta > 0\) and a weakly open set \(U\) containing \(u_0\) such that

\[
|\phi(f(t, u) - f(t_0, u_0))| < \epsilon
\]

whenever \(|t - t_0| < \delta\) and \(u \in U\).

Now, we have the following fixed point theorem, due to O’Regan, in the reflexive Banach space (see [18]) and some propositions which will be used in the sequel (see [12]).

**Theorem 1.** Let \(E\) be a Banach space and let \(Q\) be a nonempty, bounded, closed and convex subset of \(C[I, E]\) and let \(F: Q \rightarrow Q\) be a weakly sequentially continuous and assume that \(FQ(t)\) is relatively weakly compact in \(E\) for each \(t \in I\). Then, \(F\) has a fixed point in the set \(Q\).

**Proposition 1.** A convex subset of a normed space \(E\) is closed if and only if it is weakly closed.

**Proposition 2.** A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.

**Proposition 3.** Let \(E\) be a normed space with \(y \in E\) and \(y \neq 0\). Then there exists a \(\phi \in E^*\) with \(\|\phi\| = 1\) and \(\|y\| = \phi(y)\).

2. **Solvability of Volterra-Stieltjes operator**

In this section we discuss the existence of weak solutions of the equation (1) in the reflexive Banach space \(E\).

Let \(f : I \times E \rightarrow E\), \(g : I \times I \rightarrow R\) be functions such that:

(i) \(p \in C[I, E]\).
(ii) \(f : I \times E \rightarrow E\) is weakly-weakly continuous function.
(iii) There exists a constant \(M\) such that \(\|f(t, x)\| \leq M\).
(iv) The functions \(t \rightarrow g(t, t)\) and \(t \rightarrow g(t, 0)\) are continuous on \(I\).
(v) For all \(t_1, t_2 \in I\) such that \(t_1 < t_2\) the function \(s \rightarrow g(t_2, s) - g(t_1, s)\) is nondecreasing on \(I\).
(vi) \(g(0, s) = 0\) for any \(s \in I\).

**Remark 1.** Observe that Assumptions (v) and (vi) imply that the function \(s \rightarrow g(t, s)\) is nondecreasing on the interval \(I\), for any fixed \(t \in I\) (Remark 1 in [13]). Indeed, putting \(t_2 = t\), \(t_1 = 0\) in (v) and keeping in mind (vi), we obtain the desired conclusion. From this observation, it follows immediately that, for every \(t \in I\), the function \(s \rightarrow g(t, s)\) is of bounded variation on \(I\).

**Definition 2.** By a weak solution to (1) we mean a function \(x \in C[I, E]\) which satisfies the integral equation (1). This is equivalent to finding \(x \in C[I, E]\) with

\[
\phi(x(t)) = \phi(p(t) + \int_0^t f(s, x(s)) \, ds, g(t, s)), \quad t \in I \quad \forall \phi \in E^*.
\]
Now we can prove the following theorem.

**Theorem 2.** Under the assumptions (i)-(vi), the Volterra-Stieltjes integral equation (1) has at least one weak solution $x \in C[I, E]$.

**Proof.** Define the nonlinear Volterra-Stieltjes integral operator $A$ by

$$Ax(t) = p(t) + \int_0^t f(s, x(s)) \, ds, \quad t \in I.$$  

For every $x \in C[I, E], f(., x(.))$ is weakly continuous (19). To see this we equip $E$ and $I \times E$ with weak topology and note that $t \mapsto (t, x(t))$ is continuous as a mapping from $I$ into $I \times E$, then $f(., x(.))$ is a composition of this mapping with $f$ and thus for each weakly continuous $x : I \to E$, $f(., x(.)) : I \to E$ is weakly continuous, means that $\phi(f(., x(.)))$ is continuous, for every $\phi \in E^*$, $g$ is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on $I$ with respect to $s \to g(t, s)$. Thus $A$ makes sense.

Now, define the set $Q$ by

$$Q = \{ x \in C[I, E] : \| x \|_0 \leq M_0, \| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + M \| g(t_2, t_2) - g(t_1, t_1) \| + \| g(t_2, 0) - g(t_1, 0) \|, \text{ for all } t_1, t_2 \in I \}.$$  

For notational purposes $\| x \|_0 = \sup_{t \in I} \| x(t) \|.$

The remainder of the proof will be given in four steps.

**Step 1:** The operator $A$ maps $C[I, E]$ into $C[I, E]$.

Let $t_1, t_2 \in I$, $t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$

$$\| Ax(t_2) - Ax(t_1) \| \leq \| \phi(p(t_2) - p(t_1)) \| + \| \int_0^{t_2} \phi(f(s, x(s))) \, ds \| + \int_0^{t_1} \phi(f(s, x(s))) \, ds, \| \| \int_0^{t_2} \phi(f(s, x(s))) \, ds \| + \int_0^{t_1} \phi(f(s, x(s))) \, ds \| \| g(t_2, t_2) - g(t_1, t_1) \| + \| g(t_2, 0) - g(t_1, 0) \|.$$
Step 2: The operator $A$ maps $Q$ into $Q$.

Take $x \in Q$, note that the inequality (2) shows that $AQ$ is norm continuous. Then by using Proposition 3 we get

$$
\| Ax(t) \| = \| \phi(Ax(t)) \| + | \phi(\int_0^t f(s, x(s)) \, dsAx(t, s)) | \\
\leq \| p(0) + \int_0^t | \phi(f(s, x(s))) | \, dsAx(t, s) | \\
\leq \| p(0) + M \int_0^t dsAx(t, s) | \\
\leq \| p(0) + M \int_0^t \sup_{t \in I} | g(t, t) | + \sup_{t \in I} | g(t, 0) | \\
\leq \| p(0) + M \left[ k_1 + k_2 \right] = M_0,
$$

where $k_1 = \sup_{t \in I} | g(t, t) |; \; k_2 = \sup_{t \in I} | g(t, 0) |$.

Then

$$
\| Ax \| = \sup_{t \in I} \| Ax(t) \| \leq M_0.
$$

Hence, $Ax \in Q$ and $AQ \subset Q$ which prove that $A : Q \rightarrow Q$, and $AQ$ is bounded in $C[I, E]$.

Step 3: $AQ(t)$ is relatively weakly compact in $E$.

Note that $Q$ is nonempty, closed, convex and uniformly bounded subset of $C[I, E]$ and $AQ$ is bounded in norm. According to propositions [1] and [2] $AQ$ is relatively weakly compact in $C[I, E]$ implies $AQ(t)$ is relatively weakly compact in $E$, for each $t \in I$. 

Hence, $x \in Q$ and $AQ \subset Q$ which prove that $A : Q \rightarrow Q$, and $AQ$ is bounded in $C[I, E]$.
Step 4: The operator $A$ is weakly sequentially continuous.

Let $\{x_n(t)\}$ be sequence in $Q$ weakly convergent to $x(t)$ in $E$, since $Q$ is closed we have $x \in Q$. Fix $t \in I$, since $f$ satisfies (ii), then we have $f(t,x_n(t))$ converges weakly to $f(t,x(t))$. By the Lebesgue dominated convergence theorem (see assumption (iii)) for Pettis integral ([13]), we have for each $\phi \in E^*$.

$$
\phi\left( \int_0^t f(s,x_n(s)) \, ds \right) = \int_0^t \phi(f(s,x_n(s))) \, ds \, g(t,s)
$$

$$
\rightarrow \int_0^t \phi(f(s,x(s))) \, ds \, g(t,s), \quad \forall \phi \in E^*, \ t \in I.
$$

i.e. $\phi(Ax_n(t)) \rightarrow \phi(Ax(t)), \ \forall \ t \in I, \ Ax_n(t)$ converging weakly to $Ax(t)$ in $E$.

Thus, $A$ is weakly sequentially continuous on $Q$.

Since all conditions of Theorem [1] are satisfied, then the operator $A$ has at least one fixed point $x \in Q$ and the nonlinear Stieltjes integral equation (1) has at least one weak solution.

Corollary 1. Under the assumptions of Theorem 2 (with $g(t,s) = g(s)$), the Volterra-Stieltjes integral equation

$$
x(t) = p(t) + \int_0^t f(s,x(s)) \, dg(s),
$$

has a weak solution $x \in C[I,E]$.

Now, let $r > 0$ be given and define the set

$$
B_r = \{ x \in C[I,E], \ x(t) \in E : \| x \|_0 \leq r \}.
$$

Lemma 1.

Let $f : I \times B_r \rightarrow E$ be weakly-weakly continuous, then

- For each $t \in I$, $f(t,.)$ is weakly continuous, hence weakly sequentially continuous (see [3]),
- For each weakly continuous $x : I \rightarrow B_r$, $f(.,x(\cdot))$ is weakly continuous on $I$ (see [21]),
- $f$ is norm bounded, i.e., there exists an $M_r$ such that $\| f(t,x) \| \leq M_r$ for all $(t,x) \in I \times B_r$ (see [21]).

Now we have the following Theorem.

Theorem 3. Under the assumptions (i) and (iv)-(vi), if $f : I \times B_r \rightarrow E$ is weakly-weakly continuous and $M_r < r$, where $M_r$ is defined as in Lemma 4, then the Volterra-Stieltjes integral equation (4) has at least one weak solution $x \in C[I,E]$.

Proof. Define the nonlinear Volterra-Stieltjes integral operator $A$ by

$$
Ax(t) = p(t) + \int_0^t f(s,x(s)) \, ds \, g(t,s), \ t \in I.
$$

For any $x \in C[I,E]$, we have $f(.,x(.))$ is weakly continuous on $I$ (Lemma 1), then $\phi(f(.,x(.)))$ is continuous on $I$ for every $\phi \in E^*$ and hence $\phi(f(.,x(.)))$ is Riemann-Stieltjes integrable on $I$ with respect to $s \rightarrow g(t,s)$. Thus $A$ makes sense.
Now, define the set $Q$ by

$$Q = \{ x \in B_r, \| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| + M_r \{ g(t_2, t_2) - g(t_1, t_1) + | g(t_2, 0) - g(t_1, 0) | \}, \text{ for all } t_1, t_2 \in I \}. $$

For notational purposes $\| x \|_0 = \sup_{t \in I} \| x(t) \|$.

The rest of proof runs as in proof of Theorem 2.

**References**


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