INCLUSION RELATIONS FOR CERTAIN CLASS OF MULTIVALENT MEROMORPHIC FUNCTIONS

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Abstract. The purpose of the present paper is to introduce new subclasses of meromorphic multivalent functions defined by using a linear operator and obtain some inclusion relationship.

1. Introduction

Let $\Sigma_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (m, p \in \mathbb{N}),$$

which are analytic and $p-$valent in the punctured unit disk

$$D = \{ z \in \mathbb{C} : 0 < |z| < 1 \} = E \setminus \{0\},$$

where $E$ is the open unit disk.

Let $P_k(\rho)$ be the class of analytic functions $p(z)$ defined in unit disc $E = D \cup \{0\}$, satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\Re p(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi,$$

where $z = re^{i\theta}$, $k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [3]. For $\rho = 0$, we obtain the class $P_k$ defined and studied in [4], and for $\rho = 0$, $k = 2$, we get the well-known class $P$ of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater then $\rho$.

From (1.2) we can easily deduce that $p(z) \in P_k(\rho)$ if, and only if, there exist $p_1, p_2 \in P(\rho)$ such that, for $E$,

$$p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z)$$

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Let $f(z)$ is given by (1.1) and
\[ g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} b_n z^{n-p}. \] (1.4)

Then the Hadamard product (or convolution) is defined by
\[ (f \ast g)(z) = \frac{1}{z^p} + \sum_{n=m}^{\infty} a_nb_n z^{n-p} = (g \ast f)(z). \] (1.5)

In the recent paper, Noor [3] (see also [8]) introduced the following family of integral operators defined on the meromorphic functions of the class $\Sigma_p$.

Let \( q\mathcal{F}_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) be a function given by
\[ q\mathcal{F}_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \frac{1}{z^p} q\mathcal{F}_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \] (1.6)

\( q \leq s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in D, a_i, b_j \in C \setminus \mathbb{Z}^+_0; \mathbb{Z}^+_0 = \{0, -1, \ldots\}, i = 1, \ldots, q \) and \( j = 1, \ldots, s \)

where \( q\mathcal{F}_s(z) \) is the well-known generalized hypergeometric function [7].

Corresponding to \( q\mathcal{F}_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) defined by (1.6), we introduce a function
\[ q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \] by
\[ q\mathcal{F}_s(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ast q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \frac{1}{z^{p(1 - z)^{\lambda+p}}} (\lambda > -p), \] (1.7)

Therefore the function \( q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) has the following form
\[ q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(\lambda + p)_n(b_1)_n \ldots (b_s)_n}{(a_1)_n \ldots (a_q)_n} z^{n-p}. \] (1.8)

We now define the linear operator
\[ qI_s^{\lambda,p}(a_i; b_j) : \Sigma_p \to \Sigma_p. \]
by
\[ (qI_s^{\lambda,p}(a_i; b_j)f)(z) = (qI_s^{\lambda,p}(a_1, \ldots, a_q; b_1, \ldots, b_s)f)(z) = \left(q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \ast f \right)(z) \] (1.9)

\( q \leq s + 1, q, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in D, a_i, b_j \in C \setminus \mathbb{Z}^+_0; \mathbb{Z}^+_0 = \{0, -1, \ldots\}, i = 1, \ldots, q \) and \( j = 1, \ldots, s \)

Therefore the function \( q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) \) has the following form
\[ q\mathcal{F}_s^{(-1)}(a_1, \ldots, a_q; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(\lambda + p)_n(b_1)_n \ldots (b_s)_n}{(a_1)_n \ldots (a_q)_n} z^{n-p}. \] (1.10)
Thus from (1.9), we have
\[
(qI_s^{\lambda,p}(a_i; b_j)f)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{(\lambda + p)^n(b_1)_n \cdots (b_s)_n}{(a_1)_n \cdots (a_q)_n} a_n z^{n-p}. \tag{1.11}
\]

For convenience, we use the notation
\[
(qI_s^{\lambda,p}(a_i + m; b_j + n)f)(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} \frac{(\lambda + p)^n(b_1)_n \cdots (b_s)_n}{(a_1)_n \cdots (a_i + m)_n \cdots (a_q)_n} a_n z^{n-p}.
\]

(i = 1, ..., q and j = 1, ..., s)

Obviously the operators studied recently by Noor [3] and Yuan et al. [9] are special cases of \(qI_s^{\lambda,p}\) operator defined by (1.11).

It can easily be verified that
\[
z[(qI_s^{\lambda,p}(a_i + 1; b_j)f)(z)]' = a_i (qI_s^{\lambda,p}(a_i; b_j)f)(z) - (a_i + p) (qI_s^{\lambda,p}(a_i + 1; b_j)f)(z),
\]
and
\[
z[(qI_s^{\lambda,p}(a_i; b_j)f)(z)]' = (\lambda + p) (qI_s^{\lambda+1,p}(a_i; b_j)f)(z) - (\lambda + 2p) (qI_s^{\lambda,p}(a_i; b_j)f)(z).
\]

**Definition 1.1.** Let \(f \in \Sigma_p\). Then \(f \in qT_{s}^{\lambda,p,k}(\rho, \beta, a_i, b_j)\) if and only if
\[
[(1 - \beta)z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) + \beta z^p(qI_s^{\lambda+1,p}(a_i; b_j)f)(z)] \in P_k(\rho), \quad z \in E,
\]
where \(\beta > 0, k \geq 2, 0 \leq \rho < 1, \lambda > -p, p \in N\) and conditions given with (1.6) hold.

**Definition 1.2.** Let \(f \in \Sigma_p\). Then \(f \in q\Sigma S_{s}^{\lambda,p,k}(\rho, \beta, a_i, b_j)\) if and only if
\[
[\beta z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) + (1 - \beta)z^p(qI_s^{\lambda,p}(a_i + 1; b_j)f)(z)] \in P_k(\rho), \quad z \in E,
\]
where \(\beta > 0, k \geq 2, 0 \leq \rho < 1, \lambda > -p, p \in N\) and conditions given with (1.6) are satisfied.

**Lemma 1.1.** (see [5]). If \(p(z)\) is analytic in \(E\) with \(p(0) = 1\) and \(\alpha\) is a complex number satisfying \(\text{Re} (\alpha) \geq 0 (\alpha \neq 0)\), then
\[
\text{Re} [p(z) + \alpha z p' (z)] > \gamma \quad (0 \leq \gamma < 1)
\]
implies
\[
\text{Re} [p(z)] > \gamma + (1 - \gamma)(2\sigma - 1).
\]

where \(\sigma\) is given by
\[
\sigma = \sigma_{\text{Re} \alpha} = \int_{0}^{1} \left(1 + t^{\text{Re} (\alpha)}\right)^{-1} dt.
\]

**Lemma 1.2.** (see [6]). Let \(c > 0, \lambda > 0, \rho < 1\) and \(p(z) = 1 + b_1 z + b_2 z^2 + \ldots\) be analytic in \(E\). Let \(\text{Re} [p(z) + \lambda c z p' (z)] > \rho\) in \(E\), then
\[
\text{Re} [p(z) + c z p' (z)] \geq 2\rho - 1 + \left(1 - \frac{\rho}{\lambda}\right) + 2(1 - \rho) \left(1 - \frac{1}{\lambda} \right) \frac{1}{c\lambda} \int_{0}^{1} \frac{u^{\frac{1-\rho}{\lambda}-1}}{1+u} du.
\]
The result is sharp.
2. MAIN RESULTS

**Theorem 1.** Let $\beta > 0$, $\lambda > -p$, $0 \leq \rho < 1$, $p \in \mathbb{N}$ and let $f \in \mathcal{A}_{\lambda,p}(\rho, \beta, a_i, b_j)$. Then $z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) \in P_k(\rho_1)$, where

$$\rho_1 = \rho + (1 - \rho)(2\gamma_1 - 1), \quad (2.1)$$

and

$$\gamma_1 = \int_0^1 \left(1 + t^{\frac{\beta}{\lambda + p}}\right)^{-1} dt. \quad (2.2)$$

with the conditions given in (1.6).

**Proof.** Let

$$z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) = p(z). \quad (2.3)$$

Then $p(z)$ is analytic in $E$, after some calculations, we get

$$(1 - \beta)z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) + \beta z^p(qI_s^{\lambda+1,p}(a_i; b_j)f)(z) = p(z) + \frac{\beta}{\lambda + p}zp'(z).$$

Since $f \in \mathcal{A}_{\lambda,p}(\rho, \beta, a_i, b_j)$, therefore

$$\left\{p(z) + \frac{\beta}{\lambda + p}zp'(z)\right\} \in P_k(\rho) \quad \text{for} \quad z \in E.$$

This implies that

$$\text{Re} \left(\frac{p_i(z) + \frac{\beta}{\lambda + p}zp'_i(z)}{\lambda + p}\right) > \rho, \quad i = 1, 2,$$

using Lemma 1.1, we see that $\text{Re} \{p_i(z)\} > \rho_1$, where $\rho_1$ is given by (2.1). Consequently $p(z) \in P_k(\rho_1)$ for $z \in E$, and proof is complete.

Similarly we have

**Theorem 2.** Let $\beta > 0$, $\lambda > -p$, $0 \leq \rho < 1$, $p \in \mathbb{N}$ and let $f \in \mathcal{A}_{\lambda,p,k}(\rho, \beta, a_i, b_j)$. Then $z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) \in P_k(\rho_2)$, where

$$\rho_2 = \rho + (1 - \rho)(2\gamma_2 - 1), \quad (2.4)$$

and

$$\gamma_2 = \int_0^1 \left(1 + t^{\frac{\beta}{\lambda + p}}\right)^{-1} dt. \quad (2.5)$$

with the conditions given in (1.6).

**Theorem 3.** Let $\beta > 0$, $\lambda > -p$, $0 \leq \rho < 1$, $p \in \mathbb{N}$ and let $f \in \mathcal{A}_{\lambda,p,k}(\rho, \beta, a_i, b_j)$. Then $z^p(qI_s^{\lambda+1,p}(a_i; b_j)f)(z) \in P_k(\rho_3)$, where

$$\rho_3 = 2\rho - 1 + \left(\frac{1 - \rho}{\beta}\right) + 2(1 - \rho) \left(1 - \frac{1}{\beta}\right) \left(\frac{\lambda + p}{\beta}\right) \int_0^1 \frac{u^{\frac{\lambda + p}{\beta} - 1}}{1 + u} du. \quad (2.6)$$

This result is sharp.

The proof of Theorem 3 is similar to Theorem 1. Here we use Lemma 1.2 instead of Lemma 1.1.

Similarly we have

**Theorem 4.** Let $\beta > 0$, $\lambda > -p$, $0 \leq \rho < 1$, $p \in \mathbb{N}$ and let $f \in \mathcal{A}_{\lambda,p,k}(\rho, \beta, a_i, b_j)$. Then $z^p(qI_s^{\lambda,p}(a_i; b_j)f)(z) \in P_k(\rho_4)$, where

$$\rho_4 = 2\rho - 1 + \left(\frac{1 - \rho}{\beta}\right) + 2(1 - \rho) \left(1 - \frac{1}{\beta}\right) \left(\frac{a_i}{\beta}\right) \int_0^1 \frac{u^{\frac{a_i}{\beta} - 1}}{1 + u} du. \quad (2.7)$$
Next we define a function
\[ F_\delta(z) = \frac{1}{\delta} z^{\frac{1}{\delta} - p} \int_0^z t^{\frac{1}{\delta} + p - 1} f(t) dt \quad (\delta > 0, f(z) \in \Sigma_p) \]  
(2.8)

Then the linear operator \( (qI^\lambda_p(a_i; b_j)F_\delta)(z) \) satisfies the following relations.
\[ z[(qI^\lambda_p(a_i+1; b_j)F_\delta)(z)]' = a_i (qI^\lambda_p(a_i; b_j)F_\delta)(z) - (a_i + p) (qI^\lambda_p(a_i+1; b_j)F_\delta)(z), \]
and
\[ z[(qI^{\lambda+1,p}_s(a_i; b_j)F_\delta)(z)]' = (\lambda + p) (qI^{\lambda+1,p}_s(a_i; b_j)F_\delta)(z) - (\lambda + 2p) (qI^\lambda_p(a_i; b_j)F_\delta)(z). \]
(2.9)

**Theorem 5.** Let \( \beta > 0, \lambda > -p, 0 \leq p < 1, \rho \in N \) and let \( f \in qT^{\lambda,p,k}_s(\rho, \beta, a_i, b_j). \) Then \( F_\delta(z) \in qT^{\lambda,p,k}_s(\rho, \lambda + p, \beta, a_i, b_j) \) for \( z \in E \), where \( \rho_1 \) is given by (2.1) and the conditions given in (1.6) hold.

**Proof.** We have
\[ (qI^\lambda_p(a_i; b_j)F_\delta)(z) = \frac{1}{\delta} z^{-\frac{1}{\delta} - p} \int_0^z t^{\frac{1}{\delta} + p - 1} (qI^\lambda_p(a_i; b_j)f)(t) dt \]  
(2.11)

Differentiating (2.11), and using the identity (2.10), we have
\[ (1 - (\lambda + p)\beta)z^p(qI^\lambda_p(a_i; b_j)F_\delta)(z) + (\lambda + p)\beta z^{p+1}(qI^{\lambda+1,p}_s(a_i; b_j)F_\delta)(z) = z^p(qI^\lambda_p(a_i; b_j)f)(z) \]

Now using Theorem 1, we obtain the required result contained in Theorem 5.

Similarly we have

**Theorem 6.** Let \( \beta > 0, \lambda > -p, 0 \leq p < 1, \rho \in N \) and let \( f \in q\Sigma S^{\lambda,p,k}_s(\rho, \beta, a_i, b_j). \) Then \( F_\delta(z) \in q\Sigma S^{\lambda,p,k}_s(\rho_2, \delta, a_i, b_j) \) for \( z \in E \), where \( \rho_2 \) is given by (2.4) and the conditions given in (1.6) hold.

**Theorem 7.** For \( 0 \leq \beta_2 < \beta_1, \lambda > -p, 0 \leq p < 1, \rho \in N, k \geq 2, \) we have
\[ qT^{\lambda,p,k}_s(\rho, \beta_2, a_i, b_j) \subset qT^{\lambda,p,k}_s(\rho, \beta_1, a_i, b_j) \]
(2.12)

with the conditions given in (1.6).

**Proof.** For \( \beta_2 = 0 \), the proof is immediate. Let \( \beta_2 > 0 \) and \( f \in qT^{\lambda,p,k}_s(\rho, \beta_1, a_i, b_j). \) Then there exist two functions \( h_1, h_2 \in P_k(\rho) \) such that, from definition 1.1 and Theorem 1,
\[ (1 - \beta_1)z^p(qI^\lambda_p(a_i; b_j)f)(z) + \beta_1 z^{p+1}(qI^{\lambda+1,p}_s(a_i; b_j)f)(z) = h_1(z) \]  
(2.13)

and
\[ z^p(qI^\lambda_p(a_i; b_j)f)(z) = h_2(z) \]  
(2.14)

Hence
\[ (1 - \beta_2)z^p(qI^\lambda_p(a_i; b_j)f)(z) + \beta_2 z^{p+1}(qI^{\lambda+1,p}_s(a_i; b_j)f)(z) = \left( \frac{\beta_2}{\beta_1} \right) h_1(z) + \left( 1 - \frac{\beta_2}{\beta_1} \right) h_2(z) \]  
(2.15)

Since the class \( P_k(\rho) \) is a convex set, it follows that the right-hand side of (2.15) belongs to \( P_k(\rho) \) and we arrive at the result (2.12).

Similarly we have

**Theorem 8.** For \( 0 \leq \beta_2 < \beta_1, \lambda > -p, 0 \leq p < 1, \rho \in N, k \geq 2 \) then
\[ q\Sigma S^{\lambda,p,k}_s(\rho, \beta_2, a_i, b_j) \subset q\Sigma S^{\lambda,p,k}_s(\rho, \beta_1, a_i, b_j) \]
with the conditions given in (1.6).
References


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