CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY S˘ AL˘ AGEAN OPERATOR

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ABSTRACT. Making use of S˘ al˘ agean differential operator, in this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in these new subclasses. Also consequences of the results are pointed out.

1. INTRODUCTION

Let \( \mathcal{A} \) denote the class of functions of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]
which are analytic in the open unit disc \( \mathbb{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). Further, by \( \mathcal{S} \) we shall denote the class of all functions in \( \mathcal{A} \) which are univalent in \( \mathbb{U} \).

Some of the important and well-investigated subclasses of the univalent function class \( \mathcal{S} \) include (for example) the class \( \mathcal{S}^*(\beta) \) of starlike functions of order \( \beta \) in \( \mathbb{U} \) and the class \( \mathcal{K}(\beta) \) of convex functions of order \( \beta \) in \( \mathbb{U} \). By definition, we have
\[
\mathcal{S}^*(\beta) := \left\{ f : f \in \mathcal{S} \text{ and } \Re \left( \frac{zf'(z)}{f(z)} \right) > \beta ; \ z \in \mathbb{U} ; \ 0 \leq \beta < 1 \right\}
\]
and
\[
\mathcal{K}(\beta) := \left\{ f : f \in \mathcal{S} \text{ and } \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \beta ; \ z \in \mathbb{U} ; \ 0 \leq \beta < 1 \right\}.
\]
It readily follows from the definitions (2) and (3) that
\[
f \in \mathcal{K}(\beta) \iff zf' \in \mathcal{S}^*(\beta).
\]

It is well known that every function \( f \in \mathcal{S} \) has an inverse \( f^{-1} \), defined by
\[
f^{-1}(f(z)) = z, \ z \in \mathbb{U}
\]

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and

\[ f(f^{-1}(w)) = w, \quad |w| < r_0(f); \quad r_0(f) \geq \frac{1}{4}, \]

where

\[ f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \]  \quad (4)

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( \mathbb{U} \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \mathbb{U} \) given by (1). Examples of functions in the class \( \Sigma \) are

\[ \frac{z}{1 - z}, \quad -\log(1 - z), \quad \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right) \]

and so on. However, the familiar Koebe function is not a member of \( \Sigma \). Other common examples of functions in \( \mathcal{S} \) such as

\[ z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1 - z^2} \]

are also not members of \( \Sigma \) (see [7, 21]).

In 1967, Lewin [8] investigated the bi-univalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie [2] conjectured that \( |a_2| \leq \sqrt{2} \). Netanyahu [12], on the other hand, showed that \( \max_{f \in \Sigma} |a_2| = \frac{3}{4} \). Brannan and Taha [4] (see also [23]) introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses \( \mathcal{S}^*(\alpha) \) and \( \mathcal{K}(\alpha) \) of starlike and convex functions of order \( \alpha \) (0 \leq \alpha < 1), respectively (see [3]). Thus, following Brannan and Taha [4] (see also [23]), a function \( f \in \mathcal{A} \) is in the class \( \mathcal{S}_\alpha^*(\alpha) \) of strongly bi-starlike of order \( \alpha \) (0 < \alpha \leq 1), if

\[ f \in \Sigma, \quad \left| \arg\left(\frac{zf'(z)}{f(z)}\right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}; \quad 0 < \alpha \leq 1 \]

and

\[ \left| \arg\left(\frac{wg'(w)}{g(w)}\right) \right| < \frac{\alpha\pi}{2}, \quad w \in \mathbb{U}; \quad 0 < \alpha \leq 1, \]

where the function \( g \) is given by

\[ g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots \]  \quad (5)

the extension of \( f^{-1} \) to \( \mathbb{U} \).

Similarly, a function \( f \in \mathcal{A} \) is in the class \( \mathcal{K}_\alpha(\alpha) \) of strongly bi-convex functions of order \( \alpha \) (0 \leq \alpha \leq 1) if

\[ f \in \Sigma, \quad \left| \arg\left(1 + \frac{zf''(z)}{f'(z)}\right) \right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{U}; \quad 0 < \alpha \leq 1 \]

and

\[ \left| \arg\left(1 + \frac{wg''(w)}{g'(w)}\right) \right| < \frac{\alpha\pi}{2}, \quad w \in \mathbb{U}; \quad 0 < \alpha \leq 1, \]

where the function \( g \) is extension of \( f^{-1} \) to \( \mathbb{U} \).

The classes \( \mathcal{S}_\alpha^*(\beta) \) and \( \mathcal{K}_\alpha(\beta) \) of bi-starlike functions of order \( \beta \) and bi-convex functions of order \( \beta \), corresponding (respectively) to the function classes \( \mathcal{S}^*(\beta) \) and \( \mathcal{K}(\beta) \) defined by (2) and (3), were also introduced analogously. For each of the function classes \( \mathcal{S}_\alpha^*(\alpha) \) and \( \mathcal{K}_\alpha(\alpha) \), Brannan and Taha [4] found non-sharp estimates on the first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) (for details see [4, 23]).
Recently, many authors investigated bounds for various subclasses of biunivalent functions ([1], [5] - [7], [9] - [11], [13], [14], [16] and [18] - [22]). But the coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1, 2\}$; $N := \{1, 2, 3, \ldots \}$ is presumably still an open problem.

In 1983, Sălăgean [17] introduced differential operator $D_k : A \rightarrow A$ defined by

\[
D_0 f(z) = f(z),
\]

\[
D_1 f(z) = D f(z) = z f'(z),
\]

\[
D^k f(z) = D(D^{k-1} f(z)) = z(D^{k-1} f(z))', \quad k \in \mathbb{N} = \{1, 2, 3, \ldots \}.
\]

We note that

\[
D^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.
\]

The object of the present paper is to introduce two new subclasses of the function class $\Sigma$ associated with Sălăgean differential operator and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class $\Sigma$.

In order to derive our main results, we have to recall here the following lemma:

**Lemma 1** [15] If $h \in \wp$ then

\[
|c_k| \leq 2 \quad \text{for each } k,
\]

where $\wp$ is the family of all functions $h$ analytic in $U$ for which

\[
\Re\{h(z)\} > 0,
\]

where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ for $z \in U$.

2. **Coefficient bounds for the function class $S_{\Sigma}^{k, \lambda}(\alpha)$**

**Definition 1** A function $f(z)$ given by (1) is said to be in the class $S_{\Sigma}^{k, \lambda}(\alpha)$ if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \arg \left( \frac{D^{k+1} f(z)}{(1 - \lambda)D^k f(z) + \lambda D^{k+1} f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad 0 \leq \lambda < 1, \quad z \in U
\]

and

\[
\left| \arg \left( \frac{D^{k+1} g(w)}{(1 - \lambda)D^k g(w) + \lambda D^{k+1} g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad 0 \leq \lambda < 1, \quad w \in U
\]

where the function $g$ is given by (5).

**Remark 1** Taking $\lambda = 0$ in the class $S_{\Sigma}^{k, \lambda}(\alpha)$, we have $S_{\Sigma}^{k, 0}(\alpha) = S_{\Sigma}^{k}(\alpha)$ and $f \in S_{\Sigma}^{k}(\alpha)$ if the following conditions are satisfied:

\[
f \in \Sigma, \quad \left| \arg \left( \frac{D^{k+1} f(z)}{D^k f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad z \in U
\]

and

\[
\left| \arg \left( \frac{D^{k+1} g(w)}{D^k g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1; \quad w \in U
\]

where the function $g$ is given by (5).
We note that for \( k = 0 \) and \( \lambda = 0 \) the class \( S_{\Sigma}^{0,0}(\alpha) = S_{\Sigma}(\alpha) \) is class of strongly bi-starlike functions of order \( \alpha(0 < \alpha \leq 1) \). When \( k = 1 \) and \( \lambda = 0 \) the class \( S_{\Sigma}^{1,0}(\alpha) = K_{\Sigma}(\alpha) \) is class of strongly bi-convex functions of order \( \alpha(0 < \alpha \leq 1) \). For \( k = 0 \) the class was introduced and studied in [11].

We begin by finding the estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the class \( S_{\Sigma}^{k,\lambda}(\alpha) \).

**Theorem 1** Let \( f(z) \) given by (1) be in the class \( S_{\Sigma}^{k,\lambda}(\alpha) \), \( 0 < \alpha \leq 1 \) and \( 0 \leq \lambda < 1 \). Then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}} \tag{11}
\]

and

\[
|a_3| \leq \frac{\alpha}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}. \tag{12}
\]

**Proof.** It follows from (7) and (8) that

\[
\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)} = [p(z)]^\alpha
\]

and

\[
\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)} = [q(w)]^\alpha
\]

where \( p(z) \) and \( q(w) \) in \( \varphi \) and have the forms

\[
p(z) = 1 + p_1 z + p_2 z^2 + \ldots
\]

and

\[
q(z) = 1 + q_1 w + q_2 w^2 + \ldots.
\]

Now, equating the coefficients in (13) and (14), we get

\[
2^k(1-\lambda)a_2 = \alpha p_1
\]

\[
2^k(\lambda^2 - 1)a_2 + 3^k(2 - 2\lambda)a_3 = \frac{1}{2} [\alpha(\alpha - 1)p_1^2 + 2\alpha p_2]
\]

\[
- 2^k(1-\lambda)a_2 = \alpha q_1
\]

and

\[
2(1-\lambda)(2a_2^2 - a_3)3^k + (\lambda^2 - 1)2^{2k} a_2^2 = \frac{1}{2} [\alpha(\alpha - 1)q_1^2 + 2\alpha q_2].
\]

From (17) and (19), we get

\[
p_1 = -q_1
\]

and

\[
2^{2k+1}(1-\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2).
\]

From (18), (20) and (22), we obtain

\[
a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}.
\]

Applying Lemma 1 for the coefficients \( p_2 \) and \( q_2 \), we immediately have

\[
|a_2| \leq \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}}.
\]

This gives the bound on \(|a_2|\) as asserted in (11).
Next, in order to find the bound on $|a_3|$, by subtracting (20) from (18), we get
\[ 3^k(4 - 4\lambda)a_3 - 3^k(4 - 4\lambda)a_2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2). \] (23)
It follows from (21), (22) and (23) that
\[ a_3 = \frac{\alpha(p_2 - q_2)}{3^k(4 - 4\lambda)} + \frac{\alpha^2(p_1^2 + q_1^2)}{2^{2k+1}(1 - \lambda)^2}. \] (24)
Applying Lemma 1 once again for the coefficients $p_1$, $p_2$, $q_1$ and $q_2$, we readily get
\[ |a_3| \leq \frac{\alpha}{3^k(1 - \lambda)} + \frac{4\alpha^2}{2^{2k}(1 - \lambda)^2}. \]
This completes the proof of Theorem 1.

Taking $\lambda = 0$ in Theorem 1, we obtain the following corollary.

**Corollary 1** Let $f(z)$ given by (1) be in the class $S_\Sigma^0(\alpha)$ and $0 < \alpha \leq 1$. Then
\[ |a_2| \leq \frac{2\alpha}{\sqrt{4\alpha3^k + (1 - 3\alpha)2^{2k}}} \] (25)
and
\[ |a_3| \leq \frac{4\alpha^2}{2^{2k}} + \frac{\alpha}{3^k}. \] (26)

Putting $k = 0$ in Corollary 1, we obtain the coefficient estimates for well-known class $S_{\Sigma}^{0,0}(\alpha) = S_\Sigma^0(\alpha)$ of strongly bi-starlike functions of order $\alpha$ as in [4]. Considering $k = 1$ in Corollary 1, we obtain well-known class $S_{\Sigma}^{1,0}(\alpha) = K_\Sigma(\alpha)$ of strongly bi-convex functions of order $\alpha$ and coincide with results in [4].

### 3. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $M_{\Sigma}^{k,\lambda}(\beta)$

**Definition 2** A function $f(z)$ given by (1) is said to be in the class $M_{\Sigma}^{k,\lambda}(\beta)$ if the following conditions are satisfied:
\[ f \in \Sigma, \Re \left( \frac{D^{k+1}f(z)}{(1 - \lambda)D^{k}f(z) + \lambda D^{k+1}f(z)} \right) > \beta, \quad 0 \leq \beta < 1; \quad 0 \leq \lambda < 1, \quad z \in \mathbb{U} \] (27)
and
\[ \Re \left( \frac{D^{k+1}g(w)}{(1 - \lambda)D^{k}g(w) + \lambda D^{k+1}g(w)} \right) > \beta, \quad 0 \leq \beta < 1; \quad 0 \leq \lambda < 1, \quad w \in \mathbb{U}, \] (28)
where the function $g$ is given by (5).

**Remark 2** Taking $\lambda = 0$ in the class $M_{\Sigma}^{k,\lambda}(\beta)$, we have $M_{\Sigma}^{k,0}(\beta) = M_{\Sigma}^{k}(\beta)$ and $f \in M_{\Sigma}^{k}(\beta)$ if the following conditions are satisfied:
\[ f \in \Sigma, \Re \left( \frac{D^{k+1}f(z)}{D^{k}f(z)} \right) > \beta, \quad 0 \leq \beta < 1; \quad z \in \mathbb{U} \] (29)
and
\[ \Re \left( \frac{D^{k+1}g(w)}{D^{k}g(w)} \right) > \beta, \quad 0 \leq \beta < 1; \quad w \in \mathbb{U}, \] (30)
where the function $g$ is given by (5).

We note that for $k = 0$, $\lambda = 0$ the class $M_{\Sigma}^{0,0}(\beta) = S_{\Sigma}(\beta)$ is class of bi-starlike functions of order $\beta(0 \leq \beta < 1)$. When $k = 1$, $\lambda = 0$ the class $M_{\Sigma}^{1,0}(\beta) = K_{\Sigma}(\beta)$
is class of bi-convex functions of order \( \beta(0 \leq \beta < 1) \). For \( k = 0 \) the class was introduced in [11].

Next, we find the estimates on the coefficients \(|a_2|\) and \(|a_3|\) for functions in the class \( \mathcal{M}_{2; \lambda}^{k, \lambda}(\beta) \).

**Theorem 2** Let \( f(z) \) given by (1) be in the class \( \mathcal{M}_{2; \lambda}^{k, \lambda}(\beta) \), \( 0 \leq \beta < 1 \) and \( 0 \leq \lambda < 1 \). Then

\[
|a_2| \leq \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1) + 2(1-\lambda)3^k}} \tag{31}
\]

and

\[
|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2} + \frac{(1-\beta)}{3^k(1-\lambda)} \tag{32}
\]

**Proof.** It follows from (27) and (28) that there exists \( p, q \in \wp \) such that

\[
\frac{D^{k+1}f(z)}{(1-\lambda)D^k f(z) + \lambda D^{k+1} f(z)} = \beta + (1-\beta)p(z) \tag{33}
\]

and

\[
\frac{D^{k+1}g(w)}{(1-\lambda)D^k g(w) + \lambda D^{k+1} g(w)} = \beta + (1-\beta)q(w), \tag{34}
\]

where \( p(z) \) and \( q(w) \) have the forms (15) and (16), respectively. Equating coefficients in (33) and (34), we get

\[
2^k(1-\lambda)a_2 = (1-\beta)p_1 \tag{35}
\]

\[
2^{2k}(\lambda^2-1)a_2^2 + 3^k(2-2\lambda)a_3 = (1-\beta)p_2 \tag{36}
\]

\[
-2^k(1-\lambda)a_2 = (1-\beta)q_1 \tag{37}
\]

and

\[
2(1-\lambda)(2a_2^2 - a_3)3^k + (\lambda^2-1)2^{2k}a_2^2 = (1-\beta)q_2. \tag{38}
\]

From (35) and (37), we get

\[
p_1 = -q_1 \tag{39}
\]

and

\[
2^{2k+1}(1-\lambda)^2a_2^2 = (1-\beta)^2(p_1^2 + q_1^2). \tag{40}
\]

Also, from (36), (38) and (40), we obtain

\[
a_2^2 = \frac{(1-\beta)(p_2 + q_2)}{2^{2k+1}(\lambda^2-1) + 4(1-\lambda)3^k}. \]

Applying Lemma 1 for the coefficients \( p_2 \) and \( q_2 \), we immediately have

\[
|a_2| \leq \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1) + 2(1-\lambda)3^k}}. \]

This gives the bound on \(|a_2|\) as asserted in (31).

Next, in order to find the bound on \(|a_3|\), by subtracting (38) from (36), we get

\[
3^k(4-4\lambda)a_3 - 3^k(4-4\lambda)a_2^2 = (1-\beta)(p_2 - q_2). \tag{41}
\]

It follows from (39), (40) and (41) that

\[
a_3 = \frac{(1-\beta)^2(p_1^2 + q_1^2)}{2^{2k+1}(1-\lambda)^2} + \frac{(1-\beta)(p_2 - q_2)}{3^k(4-4\lambda)}. \tag{42}
\]
Applying Lemma 1 once again for the coefficients $p_1$, $p_2$, $q_1$ and $q_2$, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2} + \frac{(1-\beta)}{3^k(1-\lambda)}.$$  

This completes the proof of Theorem 2.

When $\lambda = 0$ in the Theorem 2, we get the following corollary.

**Corollary 2** Let $f(z)$ given by (1) be in the class $\mathcal{M}_\Sigma^k(\beta)$ and $0 \leq \beta < 1$. Then

$$|a_2| \leq \sqrt{\frac{1-\beta}{3^k-2^{2k-1}}}$$ (43) and

$$|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}} + \frac{1-\beta}{3^k}.$$ (44)

Putting $k = 0$ in Corollary 2, we have the coefficients estimates for the well-known class $\mathcal{M}_\Sigma^{0,0}(\beta) = \mathcal{S}_\Sigma(\beta)$ of bi-starlike functions of order $\beta$ as in [4]. Further, taking $k = 1$ in Corollary 2, we obtain the estimates for the well-known class $\mathcal{M}_\Sigma^{1,0}(\beta) = \mathcal{K}_\Sigma(\beta)$ of bi-convex functions of order $\beta$ and our results reduces to [4].

**Remark 3** For $k = 0$ the results obtained in this paper are coincide with the results discussed in [11]. Further, for the different choice of $k$ the results discussed in this paper would lead to many known and new results.

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